Differential Equations

Equation	Method
$\frac{dy}{dx} = f(x)$	$y(x) = \int f(x)dx + C$
$\frac{dy}{dx} = H(x,y) = f(x)g(y)$	$\int \frac{1}{g(y)}dy = \int f(x)dx$
$\frac{dy}{dx} + P(x)y = Q(x)$	Take $I = e^{\int P(x)dx}$, then multiply the differential equation by I and write it down in the form $D_x\left[y(x).e^{\int P(x)dx}\right] = Q(x)e^{\int P(x)dx}$ then integrate
Bernoulli's Equation (E): $\frac{dy}{dx} + h(x)y = f(x)y^n$ with $n \geq 2$	Substitute $u = y^{1-n}$ then differentiate y in terms of u and replace y and y' in (E) to get (E'): $\frac{du}{dx} + P(x)u = Q(x)$
If differential equation can be written in the form $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$	Substitute y = ux $\Rightarrow dy = udx + xdu$ or x = yv $\Rightarrow dx = vdy + ydv$, then transform the resulting equation into a separable equation
If the differential equation (E): $M(x,y)dx + N(x,y)dy = 0$ is exact differential equation i.e. $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Note: If it is written as $G(x,y)dx = H(x,y)dy$ rewrite it as $G(x,y)dx - H(x,y)dy = 0$ same as $M(x,y)dx + N(x,y)dy = 0$ where $M(x,y) = G(x,y)$ and $N(x,y) = -H(x,y)$	Take $M = \frac{\partial F}{\partial x}$ *and* $N = \frac{\partial F}{\partial y}$. Then do the following 1. $F = \int M(xy)dx + g(y)$ 2. $\frac{\partial F}{\partial y} = \left(\frac{\partial}{\partial y}\int M(x,y)dx\right) + g'(y)$ and evaluate $g'(y)$ 3. evaluate $g(y)$ and plug it in $F = \int M(xy)dx + g(y)$ 4. The implicit solution is F = c
If the differential equation (E): $M(x,y)dx + N(x,y)dy = 0$ is not exact, turn it into exact by multiplying it with a specific integrating factor	1. If $\frac{M_y - N_x}{N}$ is a function of x alone, then multiply (E) by $\mu(x) = e^{\int \frac{M_y - N_x}{N}dx}$ 2. If $\frac{N_x - M_y}{M}$ is a function of y alone, then multiply (E) by $\mu(y) = e^{\int \frac{N_x - M_y}{M}dy}$
Reduction of order can be used to find the general solution of a nonhomogeneous DE $a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$ whenever a solution y_1 is known	$y_2(x) = u(x)y_1(x)$ can be obtained directly by: $y_2 = y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}dx$

Homogeneous equation of the form $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots a_2 y'' + a_1 y' + a_0 y = 0$ with constant coefficients	Find the roots of characteristic equation • If roots are distinct and real then the solution is $y = c_1 e^{r_1 x} + c_2 e^{r_2 x} + c_3 e^{r_3 x} + \cdots c_n e^{r_n x}$ • If roots are repeated and real then the solution is $y = (c_1 + c_2 x + c_3 x^2 \cdots c_n x^{n-1}) e^{rx}$ • If roots are complex i.e. $r_1 = a + ib, \quad r_2 = a - ib$ Then the solution is $y = e^{ax}(c_1 \cos bx + c_2 \sin bx)$
Nonhomogeneous equation of the form $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots a_2 y'' + a_1 y' + a_0 y =$ $f(x)$	The solution has the form $y = y_c + y_p$ where y_c is the solution of associated homogeneous equation. Find y_p by method of variation of parameters or undetermined coefficients
To solve $a_2 y'' + a_1 y' + a_0 y = g(x)$ using variation of parameters divide by a_2 to get the 2nd order DE $y'' + Py' + Qy = f(x)$ and find $y_c = c_1 y_1 + c_2 y_2$	To find y_p, integrate $u_1' = W_1 / W$ and $u_2' = W_2 / W$ to get $u_1 \ \& u_2 \Rightarrow y_p = u_1 y_1 + u_2 y_2$ where W is the Wronskian $W(y_1(x), y_2(x))$, $W_1 = -y_2 f(x)$, and $W_2 = y_1 f(x)$
$f(x)$ 1. Any constant 2. $5x + 7$ 3. $3x^2 - 2$ 4. $x^3 - x + 1$ 5. $\sin 4x$ 6. $\cos 4x$ 7. e^{5x} 8. $(9x - 2)e^{5x}$ 9. $x^2 e^{5x}$ 10. mix	Forms of y_p 1. A 2. $Ax + B$ 3. $Ax^2 + Bx + C$ 4. $Ax^3 + Bx^2 + Cx + D$ 5. $A\cos 4x + B \sin 4x$ 6. $A\cos 4x + B \sin 4x$ 7. Ae^{5x} 8. $(Ax + B)e^{5x}$ 9. $(Ax^2 + Bx + C)e^{5x}$ 10. *mix*
Cauchy-Euler equation of the form $a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} +$ $a_0 y = f(x)$	Find the roots of auxiliary equation • If roots m_1 and m_2 are real and distinct then the solution is $y = c_1 x^{m_1} + c_2 x^{m_2}$ • If the roots are real and equal i.e. $m_1 = m_2 = m$, then the solution is $y = c_1 x^m + c_2 x^m \ln x$ • If the roots are the conjugate pair $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$ then $y = x^\alpha [c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)]$

Cauchy-Euler $\rightarrow$ DE with cte coefficients	Substitute $t = \ln x \Rightarrow x = e^t$ differentiate w.r.t. t
When a linear DE has variable coefficients, find a solution in the form of infinite series	Use Power series method where $y = \sum_{n=0}^{\infty} C_n x^n$
Definition of Laplace of f(t)	$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)dt = \text{F(s)}$
Laplace of a power function	$L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}$
Laplace of the function, whose denominator can be written in factors.	Use partial fractions
Translation Theorem	$L\{e^{at} f(t)\} = F(s-a)$ $or \quad L^{-1}\{F(s-a)\} = e^{at} f(t)$
Convolution of two functions	$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$
Convolution Theorem	$L\{f(t) * g(t)\} = L\{f(t).L\{g(t)\}$ $or\ L^{-1}(F(s).G(s)\} = f(t) * g(t)$
Differentiation of Transforms	$L\{-t.f(t)\} = F'(s)$
Integration of transforms	$L\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} f(\sigma)d\sigma$ $or\ f(t) = tL^{-1}\left\{\int_s^{\infty} f(\sigma)d\sigma\right\}$
To solve the systems of first order linear differential equations	Write down in matrix form and use eigen value method
Eigen values	To find eigen values solve $\det[A - \lambda I] = 0$
Eigen vectors	To find an eigen vector, find V such that $[A - \lambda I]V = 0$
If eigen values are distinct, then the solution is	$X(t) = c_1 V_1 e^{\lambda_1 t} + c_2 V_2 e^{\lambda_2 t} + \cdots$
If eigen values are repeated and you can't find linearly independent eigen vectors	$Find\ W\ such\ that\ [A - \lambda I]^2 W = 0$ $if [A - \lambda I]^2 = 0\ then\ take\ W = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $Find\ V\ such\ that [A - \lambda I]W = V$ $The\ solution\ is\ X = c_1 V e^{\lambda t} + c_2 (Vt + W)e^{\lambda t}$
If the eigen values are complex $\lambda = a \pm ib$	Find the eigen vector V for $a - ib$ Take $X = Ve^{a-bi}$ then separate its real and imaginary parts The solution is a linear combination of the real and imaginary parts